Linear Programming (LP)

Some Definitions

The **solution** of an LP is a vector of $\mathbb{R}^n$ representing the values for the variables.

A solution that satisfies all the constraints (functional and sign) is a **feasible solution (FS)**. A solution that does not satisfy at least one of the constraints is a **non feasible solution (NFS)**.

The set of all the feasible solutions is the **feasible region (FR)**.

An **optimal solution (OS)** is a feasible solution that has the most favourable (the maximum or the minimum) value of the objective function (OF).

When there is more than one optimal solution the problem has **alternative optimal solutions**.

The **optimum value (Z*)** is the value of the objective function at an optimal solution.

An **activity analysis problem** (for instance, the Wyndor Glass Co. problem) in its simplest version can be defined as follows:

$i = 1,2,\ldots,m$ resources or other requirements (plants)
$j = 1,2,\ldots,n$ activities (products)
x_j level of activity j (units produced)
Z global performance measure (profit)

parameters of the model

c_j $(j = 1,2,\ldots,n)$ OF' coefficients (profit of product *j*)
b_i $(i = 1,2,\ldots,m)$ right-hand-side (capacity of plant *i*)
a_{ij} $(i = 1,2,\ldots,m;\ j = 1,2,\ldots,n)$ technical coefficient (labour hours of plant *i* per unit of product *j*)

standard form

$$Z^* = Max\ Z = \sum_{j=1}^{n} c_j x_j$$

s. to:

$$\begin{cases} \sum_{j=1}^{n} a_{ij} x_j \le b_i \quad i = 1,\ldots,m & \text{functional constraints} \\ x_j \ge 0 \quad j = 1,\ldots,n & \text{sign constraints} \end{cases}$$

Hypotheses of the LP Model

(H1) proportionality: ~~x_j^2~~

The contribution of each activity (*j*) to the value of the objective function Z is proportional to the level of the activity ($c_j x_j$).

The contribution of each activity to the left-hand-side of each functional constraint is proportional to the level of the activity ($a_{ij} x_j$).

(H2) additivity: ~~$x_j x_k$~~

Every function in an LP model is the sum of the individual contributions of the respective activities.

(H3) divisibility: $\forall j,\ x_j \in \mathbb{R}.$

(H4) certainty: The value assigned to each parameter of an LP model is assumed to be a known constant.

LP Properties

Property 1: The feasible region of an LP problem is either an empty set or a convex set.

Property 2: If the feasible region of an LP problem is nonempty and bounded then at least an optimal solution exists.

Property 3: If an LP problem has optimum then at least one of its corner point feasible solutions (CPF) is an optimal solution.

Property 4: Considering an LP problem with optimum, if a CPF has no adjacent CPF with a more favourable value for the OF, then that point is an optimal solution.

Graphical Solution Method for LP Problems

Graphical resolution of problems with two decision variables.

1º) Represent the feasible region (FR)

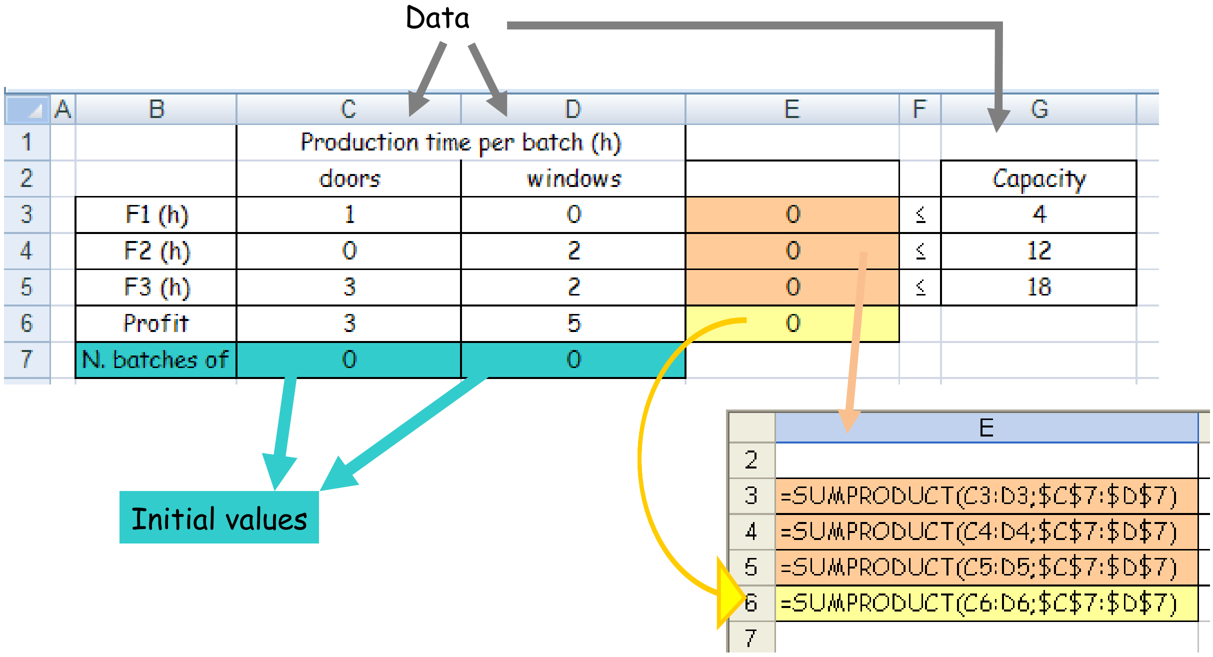
a. From the sign constraints, identify the corresponding half-plan
b. Identification of the half-plan representing each of the functional constraints (design the associate line and identify the half-plane for the related constraint)
c. Define the FR as the intersection of all the half-planes identified. If $FR = \varnothing$ the problem is infeasible, otherwise proceed.

2º) Identify, if any, the optimum (s):

a. Represent a line level of the OF (set Z to an arbitrary value) and identify the half-plane conducting to better values of Z
b. Identify the optimal solution as the corner feasible point(s) with the best value of Z, or conclude that the problem has an unbounded solution.

Solving LP Problems by the *Solver*/*Excel*

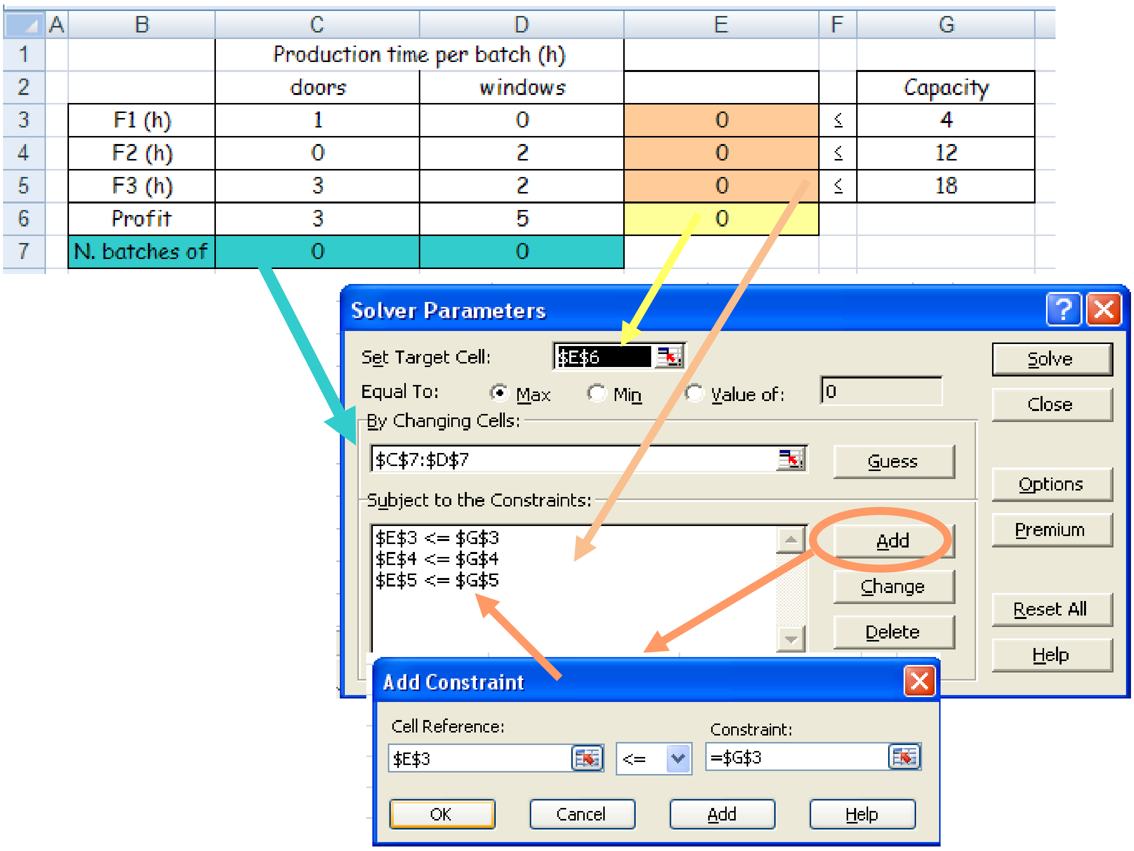
Example – Wyndor Glass Co. – the data and all formulas needed are written in an Excel spreadsheet. Create a column for each decision variable (columns “C” and “D”) and a row for each functional constraint (rows 3 - 5) and for the OF (row 6), with the respective data. Set the decision variables values to zero (row 7). Formulas in column “E” are the left-hand side of the functional constraints and the OF. The right-hand sides of the constraints are in column “G”.



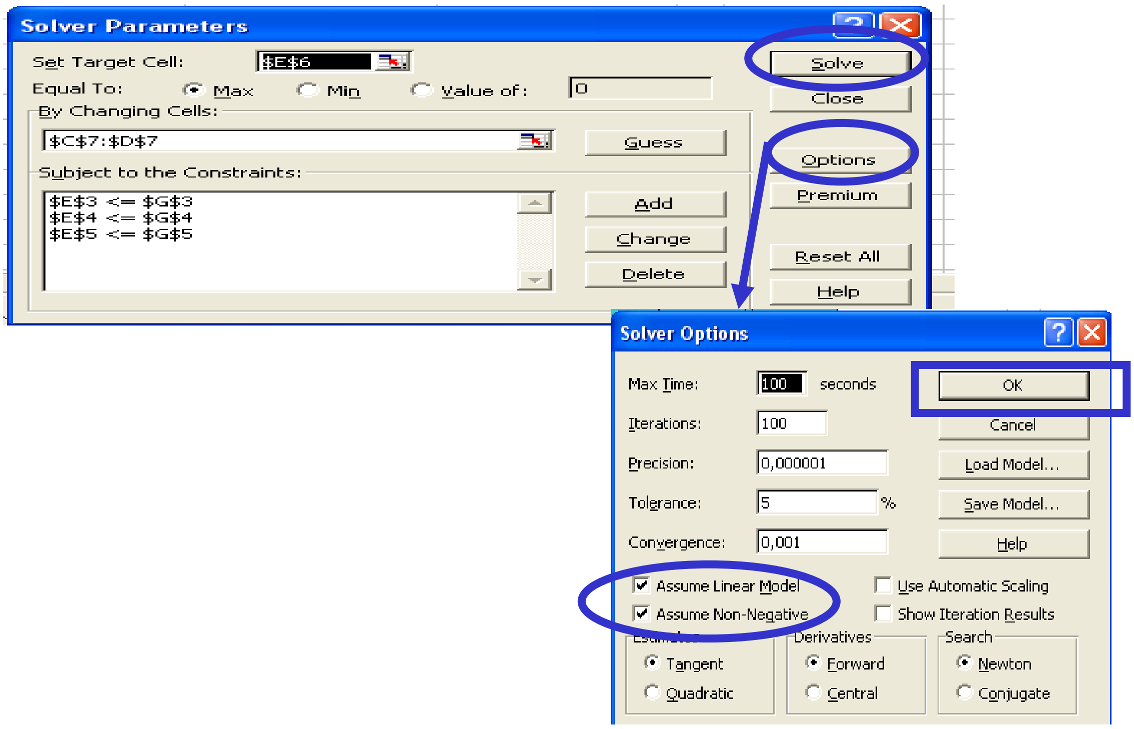
	A	B	C	D	E	F	G
1			Production time per batch (h)				
2			doors	windows			Capacity
3		F1 (h)	1	0	0	≤	4
4		F2 (h)	0	2	0	≤	12
5		F3 (h)	3	2	0	≤	18
6		Profit	3	5	0		
7		N. batches of	0	0			

	E
2	
3	=SUMPRODUCT(C3:D3;$C$7:$D$7)
4	=SUMPRODUCT(C4:D4;$C$7:$D$7)
5	=SUMPRODUCT(C5:D5;$C$7:$D$7)
6	=SUMPRODUCT(C6:D6;$C$7:$D$7)
7	

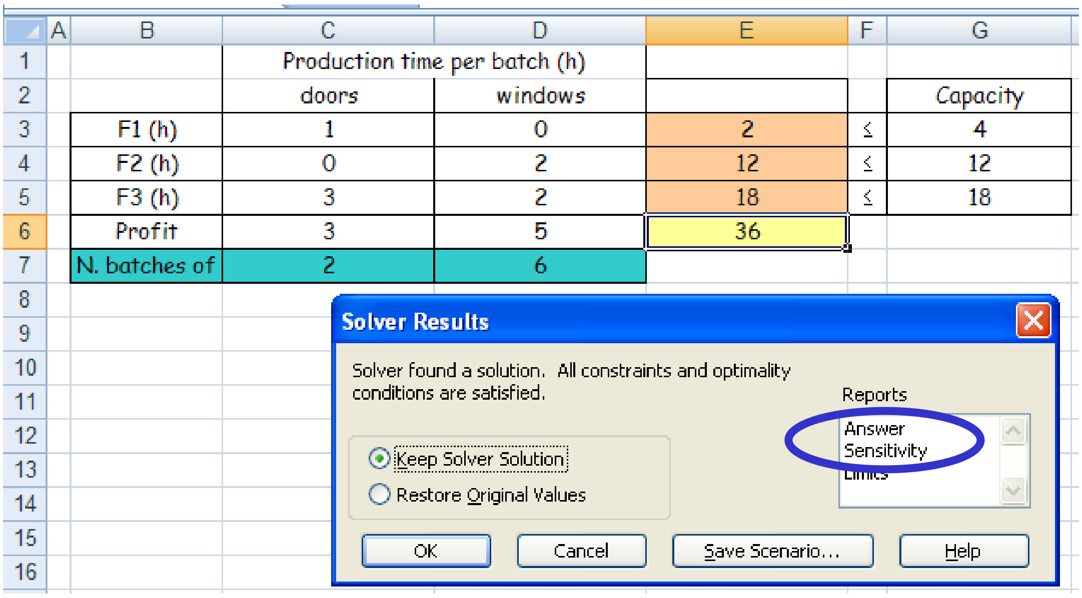
Solver – Specification of the OF cell (E6), of the objective (“Max” or “Min”) and of the cells for the decision variables values (C7:D7). Definition (“Add”) of the functional constraints (E3:E5<=G3:G5).



Solver – In the solver options, the model must be defined as a linear one with non-negative variables. To get the optimal solution the “Solve” button should be activated.



Solution – The optimal solution is then identified or it is claimed that no solution exist. If a solution exists, the output reports may be written.



Solution – Interpretation of the solution based on the *Solver* reports.

Microsoft Excel 9.0 Answer Report

Target Cell (Max)

Cell	Name	Original Value	Final Value
$E$6	Profit	0	36

→ Optimum Value

Adjustable Cells

Cell	Name	Original Value	Final Value
$C$7	N. batches of doors	0	2
$D$7	N. batches of windows	0	6

→ Optimal Solution

Constraints

Cell	Name	Cell Value	Formula	Status	Slack
$E$3	F1 (h)	2	$E$3<=$G$3	Not Binding	2
$E$4	F2 (h)	12	$E$4<=$G$4	Binding	0
$E$5	F3 (h)	18	$E$5<=$G$5	Binding	0

Microsoft Excel 9.0 Sensitivity Report

Adjustable Cells

Cell	Name	Final Value	Reduced Cost	Objective Coefficient	Allowable Increase	Allowable Decrease
$C$7	N. batches of doors	2	0	3	4,5	3
$D$7	N. batches of windows	6	0	5	1E+30	3

Constraints

Cell	Name	Final Value	Shadow Price	Constraint R.H. Side	Allowable Increase	Allowable Decrease
$E$3	F1 (h)	2	0	4	1E+30	2
$E$4	F2 (h)	12	1,5	12	6	6
$E$5	F3 (h)	18	1	18	6	6

Answer – It should be produced 2 batches of doors and 6 of windows, every week. In F1 two of the four hours are used. In F2 and F3 all the production time available per week is used, *12* and *18* hours, respectively.